Stats 2MB3, Tutorial 5

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Point Estimation

- The point estimate of a parameter θ can always be expressed by a function of all the sample data, $\hat{\theta} = f(X_1, X_2, \ldots, X_n)$.
- The unbiased estimator $\hat{\theta}$ should satisfy

$E(\hat{\theta}) = \theta$ and $E(\hat{\theta}) - \theta$ is bias.

The minimum variance unbiased estimator should satisfy $E(\hat{\theta}) = \theta$ (unbiased) and $\mathrm{Var}(\hat{\theta}) \le \mathrm{Var}(\tilde{\theta})$

(the variance is no bigger than any other estimator) .

Moment Estimate

- If $X_1, \ldots, X_n$ are random samples from a distribution, then k^{th} moment of this distribution is $\sum_{i=1}^{n} X_i^k / n$.
- If a distribution $f(\mathrm{x}; \theta_1, \ldots, \theta_m)$ has m parameters, then we should estimate them by the first m moments.
- Basically, we can obtain the moment estimates by solving the equation system $\mathrm{E}(\mathrm{X}^k) = \mathrm{g}_k(\theta_1, \ldots, \theta_m)$ where k=1,2,…m.

Ex 13, Page 254

- Consider a random sample $X_1, \ldots, X_n$ from the pdf

$$f(x;\theta) = 0.5(1+\theta x) \qquad -1 \le x \le 1$$

where $-1 \le \theta \le 1$.

Show that $\hat{\theta} = 3\overline{X}$ is an unbiased estimator of the parameter θ.

- To prove the unbiasedness, we take the expected value of the estimator

$$E(\hat{\theta}) = \mathrm{E}(3\overline{X}) = 3\,\mathrm{E}(\overline{X}) = 3\mu$$

- And $\mu = E(\mathrm{X}) = \int_{-1}^{1} x f(\mathrm{x};\theta)\,\mathrm{dx} = \int_{-1}^{1} x \frac{1+\theta x}{2}\,\mathrm{dx} = \frac{\theta}{3}$, then

$$E(\hat{\theta}) = 3\mu = 3\frac{\theta}{3} = \theta.$$

Ex 21, Page 264

- Let X have a Weibull distribution with parameters α and β, so

  $E(X)= \beta \bullet \Gamma(1+1/\alpha)$

  $Var(X)= \beta^2[\Gamma(1+2/\alpha)-(\Gamma(1+1/\alpha))^2]$.

a) Find the expressions of α and β.

b) If n=20, $\bar{x} = 28$ and $\sum x_i^2 = 16,500$, compute the estimates. $(\Gamma(1.2)^2/\Gamma(1.4)=0.95)$

- a) E(X)= β• Γ(1+1/ α)

  $E(X^2)=Var(X)+[E(X)]^2 = \beta^2\Gamma(1+2/\alpha)$

  Then

$$\overline{X} = \hat{\beta}\Gamma(1+\frac{1}{\hat{\alpha}}) \qquad (1)$$

$$\frac{\sum X_i^2}{n} = \hat{\beta}^2\Gamma(1+\frac{2}{\hat{\alpha}}) \qquad (2)$$

By (2)/(1)^2, we get $\frac{1}{n}\frac{\sum X_i^2}{\overline{X}^2} = \frac{\Gamma(1+\frac{2}{\hat{\alpha}})}{\Gamma^2(1+\frac{1}{\hat{\alpha}})}$

By (1), we have $\hat{\beta} = \frac{\overline{X}}{\Gamma(1+\frac{1}{\hat{\alpha}})}$.

- b) When n=20, $\overline{x} = 28, \sum x_i^2 = 16,500$,

$$\frac{1}{20}\left(\frac{16,500}{28^2}\right) = 1.05 = \frac{\Gamma(1+\frac{2}{\hat{\alpha}})}{\Gamma^2(1+\frac{1}{\hat{\alpha}})}$$ and its inverse

$$\frac{\Gamma^2(1+\frac{1}{\hat{\alpha}})}{\Gamma(1+\frac{2}{\hat{\alpha}})} = 0.95.$$ From the hint , $\frac{1}{\hat{\alpha}} = 0.2$ and

$$\hat{\alpha} = 5, \quad \hat{\beta} = \frac{\overline{x}}{\Gamma(1.2)} = \frac{28}{\Gamma(1.2)}$$

- EXAMPLE 3
- Consider a random sample of random variables $X_1, X_2, \ldots, X_n$ from a negative binomial population with a known parameter r and unknown parameter p.

(From page 126, the pmf is

$$nb(\mathrm{x};\mathrm{r},\mathrm{p}) = \binom{x+r-1}{r-1} p^r (1-\mathrm{p})^x \quad x=0,1,2,\ldots$$

$$E(\mathrm{X}) = \frac{r(1-\mathrm{p})}{p}$$

$$Var(\mathrm{X}) = \frac{r(1-\mathrm{p})}{p^2}$$

)

- (a) Derive the moment estimator of p.

- (b) A numerical sample of size 6 from this population was gathered, resulting in the values 7, 2, 10, 3, 5, 14 Calculate the moment estimate of p arising from this particular sample.

- Solution:
- (a)

  since p=E(X)/Var(X)=E(X)/(E(X²)-[E(X)]²), then

$$\hat{p} = \frac{\overline{X}}{\frac{\sum X_i^2}{n} - (\overline{X})^2}$$

- (b) Plug in the real numbers

$$\hat{p} = \frac{41/6}{383/6 - (41/6)^2} = 0.3987$$